Problem 1.25

Spiraling particle

A particle moves outward along a spiral. Its trajectory is given by $r = A\theta$, where A is a constant. $A = (1/\pi)$ m/rad. θ increases in time according to $\theta = \alpha t^2/2$, where α is a constant.

(a) Sketch the motion, and indicate the approximate velocity and acceleration at a few points.

(b) Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

(c) At what angles do the radial and tangential accelerations have equal magnitude?

Solution

We have

$$r(\theta) = \frac{\theta}{\pi}.$$

Substitute $\theta = \alpha t^2/2$ to make r a function of time.

$$r(t) = \frac{\alpha t^2}{2\pi}$$

As a result, the derivatives with respect to time are

$$\begin{aligned} r = \frac{\alpha t^2}{2\pi} \quad &\rightarrow \quad \dot{r} = \frac{\alpha t}{\pi} \quad \rightarrow \quad \ddot{r} = \frac{\alpha}{\pi} \\ \theta = \frac{\alpha t^2}{2} \quad &\rightarrow \quad \dot{\theta} = \alpha t \quad \rightarrow \quad \ddot{\theta} = \alpha. \end{aligned}$$

In polar coordinates the velocity vector is

$$\begin{aligned} \mathbf{v} &= \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \\ &= \frac{\alpha t}{\pi}\hat{\boldsymbol{r}} + \frac{\alpha t^2}{2\pi} \cdot \alpha t\hat{\boldsymbol{\theta}} \\ &= \frac{\alpha t}{\pi}\hat{\boldsymbol{r}} + \frac{\alpha^2 t^3}{2\pi}\hat{\boldsymbol{\theta}}. \end{aligned}$$

In polar coordinates the acceleration vector is

$$\begin{aligned} \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &= \left[\frac{\alpha}{\pi} - \frac{\alpha t^2}{2\pi}(\alpha t)^2\right]\hat{\boldsymbol{r}} + \left(\frac{\alpha t^2}{2\pi} \cdot \alpha + 2\frac{\alpha t}{\pi} \cdot \alpha t\right)\hat{\boldsymbol{\theta}} \\ &= \left(\frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi}\right)\hat{\boldsymbol{r}} + \left(\frac{\alpha^2 t^2}{2\pi} + \frac{2\alpha^2 t^2}{\pi}\right)\hat{\boldsymbol{\theta}} \\ &= \frac{\alpha}{2\pi}(2 - \alpha^2 t^4)\hat{\boldsymbol{r}} + \frac{5\alpha^2 t^2}{2\pi}\hat{\boldsymbol{\theta}}. \end{aligned}$$

Part (a)

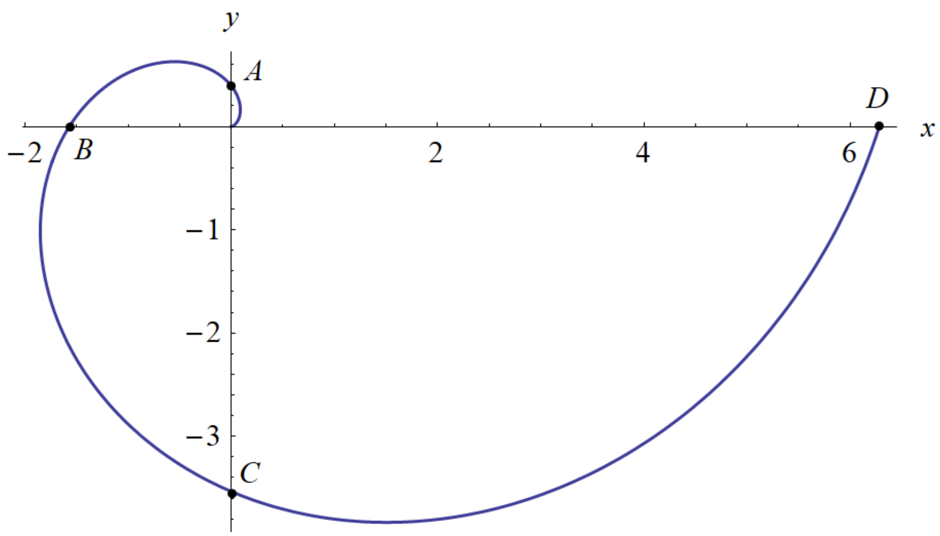

Figure 1: Plot of the particle's spiral path for $\alpha = 1$ and $0 < t < 2\pi$.

Since $\theta = \alpha t^2/2 = t^2/2$, the times at each point are

At point A:	$\frac{\pi}{2} = \frac{t_A^2}{2}$	$\rightarrow$	$t_A = \sqrt{\pi}$
At point B:	$\pi = \frac{t_B^2}{2}$	$\rightarrow$	$t_B = \sqrt{2\pi}$
At point C:	$\frac{3\pi}{2} = \frac{t_C^2}{2}$	$\rightarrow$	$t_C = \sqrt{3\pi}$
At point D:	$2\pi = \frac{t_D^2}{2}$	$\rightarrow$	$t_D = \sqrt{4\pi}.$

Therefore,

$$\text{At point } A: \quad \begin{cases} \mathbf{v}(t_A) \approx 0.56\hat{\boldsymbol{r}} + 0.89\hat{\boldsymbol{\theta}} \\ \mathbf{a}(t_A) \approx -1.3\hat{\boldsymbol{r}} + 2.5\hat{\boldsymbol{\theta}} \end{cases}$$

$$\text{At point } B: \quad \begin{cases} \mathbf{v}(t_B) \approx 0.80\hat{\boldsymbol{r}} + 2.5\hat{\boldsymbol{\theta}} \\ \mathbf{a}(t_B) \approx -6.0\hat{\boldsymbol{r}} + 5.0\hat{\boldsymbol{\theta}} \end{cases}$$

$$\text{At point } C: \quad \begin{cases} \mathbf{v}(t_C) \approx 0.98\hat{\boldsymbol{r}} + 4.6\hat{\boldsymbol{\theta}} \\ \mathbf{a}(t_C) \approx -14\hat{\boldsymbol{r}} + 7.5\hat{\boldsymbol{\theta}} \end{cases}$$

$$\text{At point } D: \quad \begin{cases} \mathbf{v}(t_D) \approx 1.1\hat{\boldsymbol{r}} + 7.1\hat{\boldsymbol{\theta}} \\ \mathbf{a}(t_D) \approx -25\hat{\boldsymbol{r}} + 10\hat{\boldsymbol{\theta}} \end{cases}.$$

Part (b)

The radial acceleration is the r-component of acceleration. Setting it equal to zero gives us

$$\frac{\alpha}{2\pi}(2-\alpha^2 t^4) = 0.$$

Solve this for t^2.

$$\begin{aligned} 2-\alpha^2 t^4 &= 0 \\ t^2 &= \frac{\sqrt{2}}{\alpha} \end{aligned}$$

This corresponds to a θ value of

$$\begin{aligned} \theta &= \frac{\alpha t^2}{2} \\ &= \frac{\alpha}{2}\cdot\frac{\sqrt{2}}{\alpha}. \end{aligned}$$

Therefore,

$$\theta = \frac{1}{\sqrt{2}} \text{ rad}.$$

Part (c)

If the radial and tangential accelerations have equal magnitudes, then

$$\left|\frac{\alpha}{2\pi}(2-\alpha^2 t^4)\right| = \left|\frac{5\alpha^2 t^2}{2\pi}\right|.$$

This equation is equivalent to these two.

$$\begin{aligned} \frac{\alpha}{2\pi}(2-\alpha^2 t^4) &= \frac{5\alpha^2 t^2}{2\pi} & \text{or} \qquad \frac{\alpha}{2\pi}(2-\alpha^2 t^4) &= -\frac{5\alpha^2 t^2}{2\pi} \\ 2-\alpha^2 t^4 &= 5\alpha t^2 & \text{or} \qquad 2-\alpha^2 t^4 &= -5\alpha t^2 \\ \alpha^2 t^4 + 5\alpha t^2 - 2 &= 0 & \text{or} \qquad \alpha^2 t^4 - 5\alpha t^2 - 2 &= 0 \\ (\alpha t^2)^2 + 5(\alpha t^2) - 2 &= 0 & \text{or} \qquad (\alpha t^2)^2 - 5(\alpha t^2) - 2 &= 0 \end{aligned}$$

Use the quadratic formula to solve for αt^2.

$$\begin{aligned} \alpha t^2 &= \frac{-5 \pm \sqrt{25+4\cdot 2}}{2} & \text{or} \qquad \alpha t^2 &= \frac{5 \pm \sqrt{25+4\cdot 2}}{2} \\ t^2 &= \frac{-5\pm\sqrt{33}}{2\alpha} & \text{or} \qquad t^2 &= \frac{5\pm\sqrt{33}}{2\alpha} \end{aligned}$$

Since t^2 has to be positive, we choose the plus sign on both of them.

$$t^2 = \frac{-5+\sqrt{33}}{2\alpha} \qquad \text{or} \qquad t^2 = \frac{5+\sqrt{33}}{2\alpha}$$

There are two values of t^2, so there are two values of θ.

$$\theta = \frac{\alpha t^2}{2} = \frac{\alpha}{2}\cdot\frac{-5+\sqrt{33}}{2\alpha} \qquad \text{or} \qquad \theta = \frac{\alpha t^2}{2} = \frac{\alpha}{2}\cdot\frac{5+\sqrt{33}}{2\alpha}$$

Therefore,

$$\theta = \frac{-5+\sqrt{33}}{4} \text{ rad} \approx 0.19 \text{ rad} \qquad \text{or} \qquad \theta = \frac{5+\sqrt{33}}{4} \text{ rad} \approx 2.7 \text{ rad}.$$